



Optimal Hedging of Uncertain Foreign Currency Returns

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SLJ Macro Partners

Fatih Yilmaz

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A. Statement of Problem

Over- or under-hedging of unknown future foreign currency cash flows induces further volatility for the underlying strategy returns for domestic currency investors, and can generate potential losses due to adverse FX rate moves. What should be the optimal hedging policy (ratio) in order to minimise risks arising from uncertain future cash flows and exchange rates?

B. Objective

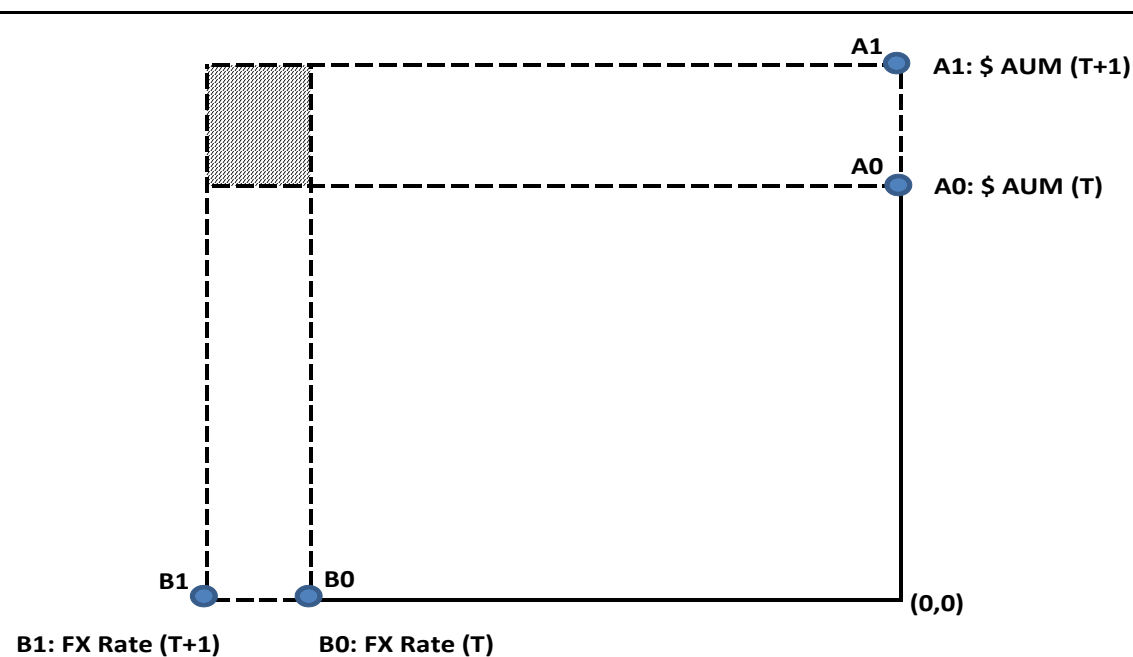
Derive an analytical solution for optimal (minimum risk) hedging policy for uncertain USD returns, from the perspective of domestic currency investors.

C. Graphical Representation of the Problem

Foreign currency returns are subject to two types of risk:

- A) Future expected returns (the gap between $A1-A0$ or expected change in AUM)
- B) Foreign exchange risk of those uncertain returns (the gap between $B1-B0$)

Exhibit 1. Optimal Hedge Ratios for Alternative Expected Return and FX Rate Changes



The shaded area in the chart above represents the problem visually. There are 4 possible cases for the shaded area:

1. Over-hedging Cash Flow and Exchange Rate Movement Generates Loss
2. Over-hedging Cash Flow and Exchange Rate Movement Generates Profit
3. Under-hedging Cash Flow and Exchange Rate Movement Generates Loss
4. Under-hedging Cash Flow and Exchange Rate Movement Generates Profit

As a concrete example, suppose that we are expecting an underlying investment strategy to generate USD100 cash flow by the end of next period, so that we hedge the whole amount by selling USD100 and buying USD100 worth of domestic currency (say SGD as an example) at today's forward rate of 1.30 (USD/SGD FX rate).

The four possible outcomes are the following:

1. We generate USD60 cash flow and USD/SGD rate moves from 1.30 to 1.50. We generate loss from over-hedging since $(60-100) \times (1.50-1.30) < 0$.
2. We generate USD60 cash flow and USD/SGD rate moves from 1.30 to 1.10. We generate profit from over-hedging since $(60-100) \times (1.10-1.30) > 0$.
3. We generate USD140 cash flow and USD/SGD rate moves from 1.30 to 1.10. We generate loss from under-hedging since $(140-100) \times (1.10-1.30) < 0$.
4. We generate USD140 cash flow and USD/SGD rate moves from 1.30 to 1.50. We generate profit from under-hedging since $(140-100) \times (1.50-1.30) > 0$.

Hence, to restate the problem, using currency forward contracts, we would like to devise an optimal hedging policy such that the risks arising from the variations in the shaded area in Exhibit 1 are minimised for domestic currency investors.

D. Optimal Hedging Ratio

The minimum variance optimal hedge ratio can be given as (derivation follows in Section F):

$$\text{Optimal Hedge Ratio} = \frac{E[r]\rho(\Delta s, \Delta f)\sigma(\Delta s) + E[\Delta s]\rho(r, \Delta f)\sigma(r)}{E[r]\sigma(\Delta f)}$$

In addition to standard minimum variance hedging solution, the decision maker can express views about:

- 1) Expected USD return of the investment strategy (i.e. expected next period cash flow)
- 2) Expected spot rate for the next period

And those two choice variables impact the optimal hedge ratio via correlation and volatility parameters.

Variable Explanations:

$E[r]$: Expected USD return for next period

$E[\Delta s]$: Expected spot FX return for next period

$\rho(\Delta s, \Delta f)$: Correlation of spot and forward FX reutrns

$\rho(r, \Delta f)$: Correlation of r and forward FX reutrns

$\sigma(\Delta s)$: Volatility of spot FX reutrns

$\sigma(\Delta f)$: Volatility of forward FX reutrns

$\sigma(r)$: Volatility of USD returns

E. Relation to Standard Minimum Variance Hedging Formula

The minimum variance hedge ratio is the ratio that minimises variance of the shaded area (in graphical representation in Exhibit 1), such that variance of profits/losses inflated due to variations in uncertain cash flows and exchange rate movements are minimised.

If expected returns or cash flows were known with perfect certainty, then volatility of returns $\sigma(r)$ would be zero. In that case, the second part in the numerator of the formula $E[\Delta s]\rho(r, \Delta f)\sigma(r) = 0$. This reduces the solution to standard optimal hedging ratio for known cash flows:

$$\text{Standard Optimal Hedge Ratio} = \frac{\rho(\Delta s, \Delta f)\sigma(\Delta s)}{\sigma(\Delta f)} = \frac{\sigma(\Delta s, \Delta f)}{\sigma^2(\Delta f)}$$

In the standard case, the optimal hedge ratio is determined simply by the normalised correlation coefficient (regression beta) of spot and forward FX returns.

On the other hand, the broader formula allows uncertainty for expected cash flows and expected FX rates to jointly enter into optimal hedging policy decision.

Optimal hedge ratio can also be decomposed as:

$$\text{Optimal Hedge Ratio} = \text{Beta}(1) + \frac{E[\Delta s]}{E[r]} \text{Beta}(2)$$

Beta(1) is the standard minimum variance hedge ratio, and it is the slope coefficient in the linear regression of spot FX returns on forward FX returns. Beta(1) measures sensitivity of spot FX returns to forward FX returns.

Similarly, Beta(2) measures the sensitivity of strategy returns to forward FX changes and its overall impact is scaled by the ratio of expected spot FX returns-to-expected returns (the ratio of the two sides of the shaded rectangle presented above).

F. Derivation of the Optimal Hedge Ratio

Let y represent net return of the hedging strategy:

$$y = r\Delta s - \lambda\Delta f$$

The terms r , Δs and Δf , respectively stand for strategy returns in foreign currency, and spot and forward exchange rate returns. The first part of the formula is the value of the shaded area in (logarithmic) graphical representation outlined in the main text. The second part of the formula is the value of the hedging policy, and λ is the hedging policy parameter. Since we are interested in finding the optimal λ that minimises the variance of y , $V(y)$, we first derive $V(y)$ as:

$$V(y) = V(r\Delta s) + \lambda^2 V(\Delta f) - 2\lambda C(r\Delta s, \Delta f)$$

In what follows, the terms V and C stand for variance and co-variance and E is the mathematical expectation operator. The terms $V(r\Delta s)$ and $C(r\Delta s, \Delta f)$ can be expanded as:

$$\begin{aligned} V(r\Delta s) &= E^2[r]V(\Delta s) + E^2[\Delta s]V(r) + E[(r - E[r])^2(\Delta s - E[\Delta s])^2] \\ &\quad + 2E[r]E[(r - E[r])(\Delta s - E[\Delta s])^2] + 2E[\Delta s]E[(r - E[r])^2(\Delta s - E[\Delta s])] \\ &\quad + 2E[r]E[\Delta s]C(r, \Delta s) - C^2(r, \Delta s) \end{aligned}$$

$$C(r\Delta s, \Delta f) = E[r]C(\Delta s, \Delta f) + E[\Delta s]C(r, \Delta f) + E[(r - E[r])(\Delta s - E[\Delta s])(\Delta f - E[\Delta f])]$$

Under multivariate normality assumption we have the following equalities:

$$2E[r]E[(r - E[r])(\Delta s - E[\Delta s])^2] = 0$$

$$2E[\Delta s]E[(r - E[r])^2(\Delta s - E[\Delta s])] = 0$$

$$[(r - E[r])^2(\Delta s - E[\Delta s])^2] = V(r)V(\Delta s) + 2C^2(r, \Delta s)$$

$$E[(r - E[r])(\Delta s - E[\Delta s])(\Delta f - E[\Delta f])] = 0$$

Hence, we can re-write variance of y as:

$$\begin{aligned} V(y) &= E^2[r]V(\Delta s) + E^2[\Delta s]V(r) + 2E[r]E[\Delta s]C(r, \Delta s) + V(r)V(\Delta s) + C^2(r, \Delta s) \\ &\quad + \lambda^2 V(\Delta f) - 2\lambda[E[r]C(\Delta s, \Delta f) + E[\Delta s]C(r, \Delta f)] \end{aligned}$$

To find the variance minimising optimal hedge parameter, we take partial derivative of $V(y)$ with respect to λ :

$$\frac{\partial V(y)}{\partial \lambda} = 2\lambda V(\Delta f) - 2[E[r]C(\Delta s, \Delta f) + E[\Delta s]C(r, \Delta f)] = 0$$

We can link λ to $E[r]$ via the optimal hedge ratio parameter β as $\lambda = \beta E[r]$. Solving for this optimal hedge ratio β , and noting that $C(\Delta s, \Delta f) = \rho(\Delta s, \Delta f)\sigma(\Delta s)\sigma(\Delta f)$ and $C(r, \Delta f) = \rho(r, \Delta f)\sigma(r)\sigma(\Delta f)$, we can derive the optimal hedge ratio as:

$$\text{Optimal Hedge Ratio} = \frac{E[r]\rho(\Delta s, \Delta f)\sigma(\Delta s) + E[\Delta s]\rho(r, \Delta f)\sigma(r)}{E[r]\sigma(\Delta f)}$$

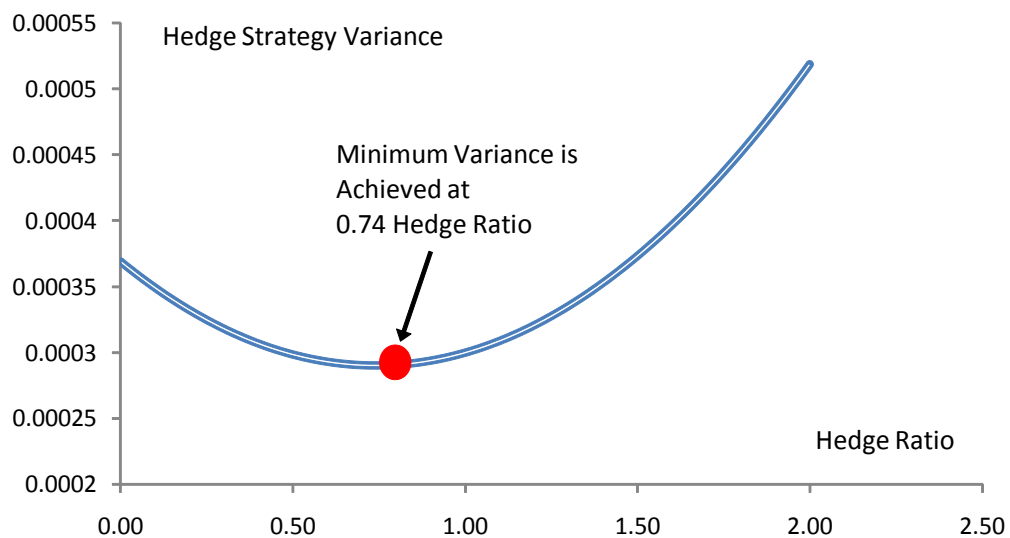
G. Numerical Example with USD/SGD

Exhibit 2. Optimal Hedge Ratios for Alternative Expected Return and USD/SGD Changes

Optimal Hedge Ratio	E[r]: Expected Annual USD Returns for TCI				Parameters	Values
	E[Δs]: Expected USD/SGD Return	10.0%	15.0%	20.0%		
-6%	1.49	1.32	1.24	1.19	ρ(Δs, Δf)	0.99
-4%	1.32	1.21	1.16	1.12	ρ(r, Δf)	-0.25
-2%	1.16	1.10	1.07	1.06	σ(r)	20.0%
0%	0.99	0.99	0.99	0.99	σ(Δf)	6.0%
2%	0.82	0.88	0.91	0.92	σ(Δs)	6.0%
4%	0.66	0.77	0.82	0.86	Beta(1)	0.99
6%	0.49	0.66	0.74	0.79	Beta(2)	-0.83

Source: SLJ Macro Partners

Exhibit 3. Example Plot of Minimum Variance – Case: E[r] = 20% and E[Δs] = 6%



Source: SLJ Macro Partners

H. Further Considerations

In this note we have concentrated on the derivation of a closed-form analytical solution for hedging uncertain returns under exchange rate uncertainty using forward rates. In order to derive an analytical solution, we concentrated purely on standard minimum variance hedging as the main objective function.

Similar problems for uncertain cash flows were also studied in the earlier literature, for instance by Rolfo (1980), Eaker and Grant (1985), and Kerkvliet and Moffet (1991).

Further extensions might consider minimising semi-variance (downside risk only) as proposed by Chen et al. (2001), or minimum VaR hedge ratio as in Harris and Shen (2006), or other suitable objective functions.

Other extensions might include incorporation of options into the analysis as in Lean and Tse (2001), or allowing for non-normal distributional effects by way of analytical expansion or numerical simulation techniques (i.e. Monte Carlo or bootstrapping).

Finally, we have not addressed the issue of calibrating covariance matrices for the optimal hedge ratio. More efficient measurement of risk and allowing for conditional variation in covariance parameters over time should in theory improve out-of-sample performance of the objective function in practice.

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